Int. J. Solids Structures Vol. 30, No. 5, pp. 675–694, 1993
Printed in Great Britain

0020–7683/93 $6.00+ .00
Pergamon Press Ltd

ON TIMOSHENKO–REISSNER TYPE THEORIES OF PLATES AND SHELLS

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(*Received* 10 *October* 1991 ; *in revised form* 6 *August* 1992)

Abstract—Linear TR theories (Timoshenko–Reissner theories) of isotropic plates and shells are discussed. These theories take into account the transversal shear deformation and rotation inertia. The main subject under consideration is the construction of these theories by the asymptotic method and the related error estimates for static and dynamic problems. In the dynamic case a method is suggested for the extension of the range of applicability of the TR theory.

1. INTRODUCTION

The asymptotic method for the derivation of the differential equations of the general theory of shells and plates [see e.g. Friedrichs and Dressler (1961), Reiss and Locke (1961), Green (1962, 1963), Aksentian and Vorovich (1963), Reissner (1964) and Goldenveizer (1966, 1976, 1980)] distinctly shows the following physically evident aspect of the stress–strain state (SSS) of thin elastic bodies (shells and plates): the overall SSS is a composition of the interior and boundary parts. This fact is reflected in the asymptotic method by the two iterative processes for the integration of the differential equations of elasticity when the domain of integration is sufficiently thin. The first of these processes allows interior integrals to be constructed (i.e. solutions with relatively slow variation, which are generally defined on the entire domain occupied by a thin elastic body) to within the given asymptotic error. The second process results in rapidly varying integrals localized near the edges and forming the so-called boundary layer (in dynamic problems boundary layer solutions sometimes turn into rapidly oscillating ones).

The interior integrals can be subjected to face conditions which express the fact that a given exterior load is applied to the top and bottom surfaces. In this case there are several arbitrary parameters to be chosen in order to satisfy certain conditions on the edge, which characterize the edge clamping or loading.

Two-dimensional theories of shells and plates can be considered as a means for the approximate construction of the interior integral. The corresponding SSS is close to the three-dimensional SSS at sufficiently large distances from the edge. The order of approximation of the interior integral corresponds to the two-dimensional theory of the particular order of accuracy. In this paper we adopt the term "asymptotic theory of type $[O(\eta^{\rho}) = 0]$" for a theory† (a system of two-dimensional equations without account of the boundary conditions) based on the use of interior integrals that were obtained while neglecting in the equations of three-dimensional elasticity the terms of order $O(\eta^{\gamma})$ with respect to the unity, provided that $\gamma \geqslant \rho$. Here $\eta = h/R$ is a small geometric parameter, $2h$ is the thickness of a thin elastic body, R is its characteristic linear dimension.

The boundary integrals play a twofold role in the general asymptotic theory. First, they allow the construction of the approximate SSS on the cross-sections containing the edges, i.e. to solve a problem not covered by any two-dimensional theory. The second aspect is that the analysis of the interaction between the interior SSS with the boundary layer allows the formulation of (generally with any given accuracy) the boundary conditions which two-dimensional theories must agree with, as well as to estimate the errors due to the approximate boundary conditions. For details see Section 4.

† We will use the notation $[O(\eta^{\rho}) = 0]$ referring to formulas as well.

It is shown in Goldenveizer (1976) for the static case that the asymptotic shell theory of type $[O(\eta^{2-2q}) = 0]$, where q is the variation index of the SSS to be found, is in essence the slightly modified Kirchhoff–Love theory (KL theory). The asymptotic theory of type $[O(\eta^{4-4q}) = 0]$ is here referred to as the asymptotic Timoshenko–Reissner theory (TR theory). As is shown below, this theory takes into account shear deformation and rotation inertia together with some other characteristics of the same asymptotic order. Theories where shear deformation and rotation inertia are admitted on the basis of physical assumptions are here referred to as engineering TR theories. The theories taking into account the transversal shear deformation and the rotation inertia have been named after Timoshenko and Reissner to be associated with the scientists whose works [see e.g. Timoshenko (1921) and Reissner (1944)] devoted to the analysis of these factors in mechanics of structures caused the greatest response.

A relatively complete bibliography on the engineering TR theories can be found in the general reference works of Ainola and Nigul (1965), Koiter and Simmonds (1973), Grigolyuk and Selezov (1973), Simmonds (1976) and Reissner (1985); see also Lo *et al.* (1977) and Reddy (1984).

Remark. In the general dynamic case we should rather speak of asymptotic theories of type $[O(\eta^{2-2q}+\eta^{2-2a}) = 0]$ and $[O(\eta^{4-4q}+\eta^{4-4a}) = 0]$ instead of $[O(\eta^{2-2q}) = 0]$ and $[O(\eta^{4-4q}) = 0]$; here a is the dynamicity index (see Section 2). However, the class of surface loads we consider below guarantees that the condition $q \geqslant a$ is satisfied by all means. Therefore, we shall speak of theories of the type $[O(\eta^{2-2q}) = 0]$ and $[O(\eta^{4-4q}) = 0]$ in both the dynamic and static cases.

The aim of this paper is to discuss, as thoroughly as possible, the TR theories from the point of view of the general asymptotic theory of plates and shells. This discussion seems justified, since the TR theories have been applied surprisingly often in modern research (e.g. to nonlinear problems, stationary and nonstationary dynamics, shells having a fairly complicated structure, shells interacting with other continuous media or physical fields). For all these applications, one needs to know the reliability of the refinements the TR theories claim to provide.

The advantages of the TR theories are far from being unquestionable. As shown below, the transition from the equations of the KL theories to those of the TR theories, at least in some cases, leads to certain refinements which turn out to be illusory. On the other hand, the TR theories have definite advantages when applied to numerous practically important problems (e.g. the problems of vibrations and wave propagation).

It is also shown in the present paper that there exist mathematical approaches to the theory of plates and shells, which provide stronger results than those obtained on the basis of the engineering TR theories as well as the asymptotic TR theories.

No attempt has been made here to give a complete bibliography on the subject, or to analyse critically the works referred to, or to discuss the priority of the results mentioned.

2. ASYMPTOTIC TR THEORY OF PLATE BENDING

Here we describe a method to obtain the asymptotic theory of isotropic plate bending. The position of the points of the plate in three-dimensional space will be determined by the radius vector

$$\mathbf{R} = \mathbf{r}(x^1, x^2) + x^3\mathbf{n},$$

where $\mathbf{r}$ is the radius vector of the midplane S, $\mathbf{n}$ is the unit normal to S, (x^1, x^2) are curvilinear coordinates on S, x^3 is the distance from S along the normal. Denote by $a_{\alpha\beta}$ the metric tensor of the midplane S (here and in what follows the Greek indices are assumed to take values 1 and 2). Let σ^{ij} $(i, j = 1, 2, 3)$ be the stress tensor, and let the displacement vector of the elastic body be given by the formula

$$\mathbf{u} = u^\alpha \mathbf{r}_\alpha + w\mathbf{n}.$$

Then the three-dimensional equations of elastodynamics can be written as:

the equations of equilibrium

$$\nabla_\alpha \sigma^{\alpha\beta} + \frac{\partial \sigma^{3\beta}}{\partial x^3} - \rho \frac{\partial^2 u^\beta}{\partial t^2} = 0,$$

$$\nabla_\alpha \sigma^{3\alpha} + \frac{\partial \sigma^{33}}{\partial x^3} - \rho \frac{\partial^2 w}{\partial t^2} = 0, \tag{1}$$

the formulas expressing the "displacement-stress law"

$$E\frac{\partial w}{\partial x^3} = \sigma^{33} - \nu a_{\lambda\mu}\sigma^{\lambda\mu},$$

$$E\left(\nabla_\alpha w + \frac{\partial u_\alpha}{\partial x^3}\right) = 2(1+\nu)a_{\lambda\alpha}\sigma^{3\lambda},$$

$$Ee_{\alpha\beta} = (1+\nu)\sigma_{\alpha\beta} - \nu a_{\alpha\beta}a^{\lambda\mu}\sigma_{\lambda\mu} - \nu a_{\alpha\beta}\sigma^{33},$$

$$e_{\alpha\beta} = \tfrac{1}{2}(\nabla_\beta u_\alpha + \nabla_\alpha u_\beta), \tag{2}$$

where ∇_α denotes the covariant derivative, ρ is the mass density of the plate, E is Young's modulus and ν is Poisson's ratio. We shall also take into account the face conditions

$$\sigma^{33}|_{x^3=\pm h} = \pm Q^3, \quad \sigma^{3\alpha}|_{x^3=\pm h} = Q^\alpha, \tag{3}$$

where Q^α and Q^3 are the tensors of tangential and normal surface loads.

Remark. It is assumed in (3) that the exterior loads are applied to both faces in such a way that the SSS of the plate can only be antisymmetrical with respect to the midplane (the SSS of bending). Exterior loads in general can produce, apart from the bending SSS, the SSS which is symmetrical with respect to the midplane (the so-called SSS of extension and transversal compression). An asymptotic method for the latter type of SSS is discussed in Section 6.

According to the asymptotic method we first dilate the scale of the independent variables setting

$$x^\alpha = R\eta^q \xi^\alpha, \quad x^3 = R\eta\zeta, \quad t = Rc_{sh}^{-1}\eta^a\tau, \tag{4}$$

where

$$c_{sh} = \sqrt{E/\rho},$$

and then we make the following:

Assumption 1. Differentiation with respect to the variables ξ^α, ζ, τ does not change the asymptotic order of the quantities to be found.

Then the numbers q and a in (4) can be given the following physical interpretation: q is the variation index of the SSS to be found with respect to the variables x^α, a is the dynamicity index, i.e. an asymptotic quantity characterizing the rate with which the processes develop in time.

Let us also assume that q and a satisfy the inequalities

$$q < 1, \quad a < 1, \tag{5}$$

which are the necessary conditions for the validity of any two-dimensional theory of shells or plates. Moreover, in this paper we shall only deal with such q, a that satisfy one of the two following inequalities

$$q \geqslant 1/2+a/2, \quad q \geqslant a. \tag{6}$$

In the case of thin elastic bodies it is shown in Goldenveizer (1987) that the first inequality (6) holds for the so-called quasi-transversal vibrations (i.e. when $|w| \gg |u^\alpha|$), and the second inequality (6) is valid for the quasi-tangential vibrations (i.e. when $|u^\alpha| \gg |w|$). The exterior loads considered in Goldenveizer (1987) were applied to the edges of the body. For some particular surface loads the results of Goldenveizer (1987) do not hold, but these cases and the related problems are not considered here [see Kaplunov and Nolde (1992b)]. In the case of plate bending we admit the first inequality (6) and, therefore, the inequality $q > a$, since $a < 1$.

Next, we introduce dimensionless displacements and stresses setting

$$w = Rw^*, \quad u_\alpha = R\eta^{1-q}u_\alpha^*, \quad \sigma^{\alpha\beta} = E\eta^{1-2q}\sigma_*^{\alpha\beta}, \quad \sigma^{3\alpha} = E\eta^{2-3q}\sigma_*^{3\alpha}, \quad \sigma^{33} = E\eta^{3-4q}\sigma_*^{33}, \tag{7}$$

and we make the following:

Assumption 2. All quantities marked by the asterisk have the form $O(\eta^\varkappa)$ (the value of $\varkappa$ is assumed to be the same for all these quantities).

Remark. Formulas (4), (7) determine the asymptotic properties of a plate subjected to bending and are in agreement with the common idea of its SSS. However, relations (4), (7) are not equivalent mathematical expressions of certain physical assumptions. These relations can also be obtained on a strict mathematical basis, but that would require too much printed space. There is also the possibility of verifying conditions (4), (7) *a posteriori*: they are used together with Assumptions 1 and 2, and one can check that a sufficiently general class of integrals of the approximate equations, which follow from (4), (7), indeed possess the assumed properties. As is shown below, some "extra" integrals appear in the asymptotic theory of type $[O(\eta^{4-4q}) = 0]$ (they lack the assumed asymptotic properties). In general, these integrals do not describe (even approximately) elastic phenomena in plates and shells, and it is natural to consider the possibility of their elimination, which shall be done later.

Let us rewrite the equilibrium equations (1) and the "displacements-stresses" formulas (2), taking into account (4), (7):

$$\begin{aligned} \frac{\partial \sigma_*^{3\beta}}{\partial \zeta} &= -\tilde{\nabla}_\alpha \sigma_*^{\alpha\beta} + \eta^{2q-2a}\frac{\partial^2 u_*^\beta}{\partial \tau^2}, \\ \frac{\partial \sigma_*^{33}}{\partial \zeta} &= -\tilde{\nabla}_\alpha \sigma_*^{3\alpha} + \eta^{4q-2a-2}\frac{\partial^2 w^*}{\partial \tau^2}, \end{aligned} \tag{8}$$

and

$$\begin{aligned} \frac{\partial w^*}{\partial \zeta} &= \eta^{4-4q}\sigma_*^{33} - \nu\eta^{2-2q}a_{\lambda\mu}\sigma_*^{\lambda\mu}, \\ \frac{\partial u_\alpha^*}{\partial \zeta} &= -\tilde{\nabla}_\alpha w^* + 2(1+\nu)\eta^{2-2q}a_{\lambda\alpha}\sigma_*^{3\lambda}, \\ \sigma_*^{\alpha\beta} &= \frac{1}{1-\nu^2}[(1-\nu)e_*^{\alpha\beta} + \nu a^{\alpha\beta}a_{\lambda\mu}e_*^{\lambda\mu}] + \frac{\nu}{1-\nu}\eta^{2-2q}a^{\alpha\beta}\sigma_*^{33}, \\ e_{\alpha\beta}^* &= \tfrac{1}{2}(\tilde{\nabla}_\beta u_\alpha^* + \tilde{\nabla}_\alpha u_\beta^*). \end{aligned} \tag{9}$$

Here $\tilde{\nabla}_\alpha = R\eta^q\nabla_\alpha$. This operator, as well as the operators $\partial/\partial\xi_1$, $\partial/\partial\xi_2$, does not change the asymptotic order of the quantities to be found.

The first inequality (6) with $a < 1$ obviously implies that

$$2q-2a \geqslant 2-2q. \tag{10}$$

Therefore, in (8), (9) all explicitly written powers of η have non-negative exponents. We can also see that if the first inequality in (6) is strict then the asymptotic contribution of the inertial terms becomes asymptotically secondary and, thus, eqns (8), (9) for $q > 1/2+a/2$ correspond to the quasi-static case (or the static case when $a \to -\infty$). If $q = 1/2+a/2$ we have the genuine dynamic case which, in particular, includes free bending vibrations.

There exists a class of three-dimensional SSSs which are polynomial in the transversal variable ζ and are described by eqns (8), (9) and the face conditions (3) with accuracy $[O(\eta^{4-4q}) = 0]$. This class is defined by the formulas:

$$\begin{aligned}
w^* &= w^{(0)}+\zeta^2\eta^{2-2q}w^{(2)},\\
u^*_\alpha &= \zeta u^{(1)}_\alpha+\zeta^3\eta^{2-2q}u^{(3)}_\alpha,\\
e^{\alpha\beta}_* &= \zeta e^{\alpha\beta}_{(1)}+\zeta^3\eta^{2-2q}e^{\alpha\beta}_{(3)},\\
\sigma^{\alpha\beta}_* &= \zeta\sigma^{\alpha\beta}_{(1)}+\zeta^3\eta^{2-2q}\sigma^{\alpha\beta}_{(3)},\\
\sigma^{3\beta}_* &= \sigma^{3\beta}_{(0)}+\zeta^2\sigma^{3\beta}_{(2)}+\zeta^4\eta^{2-2q}\sigma^{3\beta}_{(4)},\\
\sigma^{33}_* &= \zeta\sigma^{33}_{(1)}+\zeta^3\sigma^{33}_{(3)}+\zeta^5\eta^{2-2q}\sigma^{33}_{(5)},
\end{aligned} \tag{11}$$

where the quantities with an additional index in parentheses are functions of the variables ξ^1, ξ^2, τ, or, equivalently, of x^1, x^2, t, and have the form $O(\eta^\varkappa)$. These quantities are related by the following equations:

$$\begin{aligned}
u^{(1)}_\alpha &= -\tilde{\nabla}_\alpha w^{(0)}+2(1+\nu)\eta^{2-2q}a_{\lambda\alpha}\sigma^{3\lambda}_{(0)},\\
e^{(1)}_{\alpha\beta} &= \tfrac{1}{2}(\tilde{\nabla}_\beta u^{(1)}_\alpha+\tilde{\nabla}_\alpha u^{(1)}_\beta),\\
\sigma^{\alpha\beta}_{(1)} &= \frac{1}{1-\nu^2}[(1-\nu)e^{\alpha\beta}_{(1)}+\nu a^{\alpha\beta}a_{\lambda\mu}e^{\lambda\mu}_{(1)}]+\frac{\nu}{1-\nu}\eta^{2-2q}a^{\alpha\beta}\sigma^{33}_{(1)},\\
\sigma^{3\beta}_{(2)} &= -\tfrac{1}{2}\tilde{\nabla}_\alpha\sigma^{\alpha\beta}_{(1)}+\tfrac{1}{2}\eta^{2q-2a}\frac{\partial^2 u^\beta_{(1)}}{\partial\tau^2},\\
\sigma^{3\beta}_{(0)} &= -\sigma^{3\beta}_{(2)}-\eta^{2-2q}\sigma^{3\beta}_{(4)}+Q^\beta_*,\\
\sigma^{33}_{(1)} &= -\tilde{\nabla}_\alpha\sigma^{3\alpha}_{(0)}+\eta^{4q-2a-2}\frac{\partial^2 w^{(0)}}{\partial\tau^2},\\
\sigma^{33}_{(3)} &= -\tfrac{1}{3}\tilde{\nabla}_\alpha\sigma^{3\alpha}_{(2)}+\tfrac{1}{3}\eta^{2q-2a}\frac{\partial^2 w^{(2)}}{\partial\tau^2},
\end{aligned} \tag{12}$$

and

$$\sigma^{33}_{(1)}+\sigma^{33}_{(3)}+\eta^{2-2q}\sigma^{33}_{(5)} = Q^3_*, \tag{13}$$

and

$$\begin{aligned}
w^{(2)} &= -\tfrac{1}{2}\nu a_{\lambda\mu}\sigma^{\lambda\mu}_{(1)},\\
u^{(3)}_\alpha &= -\tfrac{1}{3}\tilde{\nabla}_\alpha w^{(2)}+\tfrac{2}{3}(1+\nu)a_{\lambda\alpha}\sigma^{3\lambda}_{(2)},\\
e^{(3)}_{\alpha\beta} &= \tfrac{1}{2}(\tilde{\nabla}_\beta u^{(3)}_\alpha+\tilde{\nabla}_\alpha u^{(3)}_\beta),\\
\sigma^{\alpha\beta}_{(3)} &= \frac{1}{1-\nu^2}[(1-\nu)e^{\alpha\beta}_{(3)}+\nu a^{\alpha\beta}a_{\lambda\mu}e^{\lambda\mu}_{(3)}]+\frac{\nu}{1-\nu}a^{\alpha\beta}\sigma^{33}_{(3)},\\
\sigma^{3\beta}_{(4)} &= -\tfrac{1}{4}\tilde{\nabla}_\alpha\sigma^{\alpha\beta}_{(3)},\\
\sigma^{33}_{(5)} &= -\tfrac{1}{5}\tilde{\nabla}_\alpha\sigma^{3\alpha}_{(4)},
\end{aligned} \tag{14}$$

where

$$Q_*^\alpha = E^{-1}\eta^{3q-2}Q^\alpha, \quad Q_*^3 = E^{-1}\eta^{4q-3}Q^3.$$

Here the order of the dimensionless loads Q_*^3 and Q_*^α is not larger than that of the functions marked by the asterisk in (11) [this is due to (7) and the face conditions (3)]. In order to verify the above result, we should substitute the relations (11)–(14) into eqns (8)–(9) and the face conditions (3) neglecting, in the intermediate calculations, the terms of order $O(\eta^{4-4q})$ with respect to unity.

On the basis of the relations (12)–(14) we can derive the systems of two-dimensional differential equations of the asymptotic theories of bending of plates. These theories can be either of type $[O(\eta^{2-2q}) = 0]$ or $[O(\eta^{4-4q}) = 0]$.

Now we consider the first theory, which corresponds to the KL theory. Discarding in (12), (13) the terms containing η^{2-2q} as well as the terms with η^{2q-2a} [since the inequality (10) holds], and considering the equalities (12) in the same order as they are written, one can successively express the functions

$$u_\alpha^{(1)}, e_{\alpha\beta}^{(1)}, \sigma_{(1)}^{\alpha\beta}, \sigma_{(2)}^{3\beta}, \sigma_{(0)}^{3\beta}, \sigma_{(1)}^{33}, \sigma_{(3)}^{33} \tag{15}$$

in terms of $w^{(0)}$ and the components Q^β of the exterior surface load. After substituting in (13) the above results and performing rather lengthy but simple calculations one obtains the classical differential equation of Kirchhoff for $w^{(0)}$. Further, assuming $w^{(0)}$ is known, we can find the stresses and the displacements for a three-dimensional plate after substituting the functions (15) in (11), where the terms of the above type should also be omitted. This proves that the asymptotic theory of plate bending of type $[O(\eta^{2-2q}) = 0]$ is adequate for the classical Kirchhoff theory. A more general result is established in Goldenveizer (1976): the asymptotic shell theory of type $[O(\eta^{2-2q}) = 0]$, i.e. the so-called KL theory of the Introduction, is adequate for the slightly modified two-dimensional theory based on the assumptions of the Kirchhoff–Love type.

The asymptotic theory of plate bending of type $[O(\eta^{4-4q}) = 0]$ can be derived on the basis of the following iterative process. According to the method described above, let us express the functions (15) in terms of $w^{(0)}$, Q^β, with the accuracy $[O(\eta^{2-2q}) = 0]$, using the simplified formulas (12), where the terms containing η^{2-2q} and η^{2q-2a} are neglected. Then, using (14) we successively express the functions

$$w^{(2)}, u_\alpha^{(3)}, e_{\alpha\beta}^{(3)}, \sigma_{(3)}^{\alpha\beta}, \sigma_{(4)}^{3\beta}, \sigma_{(5)}^{33} \tag{16}$$

in terms of $w^{(0)}$, Q^β. Next, for the quantities (15), we construct corrections of order η^{2-2q}, replacing in (12) the previously neglected terms with $\sigma_{(0)}^{3\beta}$, $\sigma_{(1)}^{33}$, $\sigma_{(4)}^{3\beta}$, $u_\alpha^{(1)}$, $w^{(2)}$ by their approximate expressions. As a result, the functions in (15), (16) can be written in terms of $w^{(0)}$ and Q^β with the accuracy $[O(\eta^{2-2q}) = 0]$ and $[O(\eta^{4-4q}) = 0]$, respectively. Substituting the expressions obtained in this way in the unsimplified equality (13), we obtain an equation for $w^{(0)}$ with the accuracy $[O(\eta^{4-4q}) = 0]$. In the initial independent variables x^α, t this equation can be written as

$$\frac{2Eh^3}{3(1-\nu^2)}\left[1+h^2\frac{8-3\nu}{10(1-\nu)}\Delta\right]\Delta^2 W + 2\rho h\left[1+h^2\frac{\nu-2}{6(1-\nu)}\Delta\right] \times \frac{\partial^2 W}{\partial t^2} - 2h\left[1+h^2\frac{2-\nu}{3(1-\nu)}\Delta\right]\nabla_\lambda Q^\lambda - 2Q^3 = 0. \tag{17}$$

Here $W = w|_{x^3=0} = Rw^{(0)}$, $\Delta = a^{\lambda\mu}\nabla_\lambda\nabla_\mu$ is the Laplace operator corresponding to the metric tensor of the midplane.

The relation (17) is fit to be called the resolving equation of the two-dimensional theory of plate bending of type $[O(\eta^{4-4q}) = 0]$. Each solution of this equation determines the displacements and stresses in the three-dimensional domain occupied by the plate, according to the formulas (11), where the terms with η^{2-2q} are preserved. The coefficients in (11) with powers of ζ are given by (15), (16) and can be taken as known. This fact is due to the above method used for the derivation of eqn (17). The corresponding formulas expressing the quantities (15), (16) can be easily obtained, but they are too lengthy to be written out completely. Here we give only the formula for $\sigma^{\alpha\beta}$ $(\alpha, \beta = 1, 2)$

$$\begin{aligned}\sigma^{\alpha\beta} = Ex^3\bigg\{&-\frac{1}{1-\nu^2}[(1-\nu)\nabla^\beta\nabla^\alpha W+\nu a^{\alpha\beta}\Delta W]\\&-\frac{h^2}{1-\nu^2}\left[\Delta\nabla^\beta\nabla^\alpha W+\frac{5\nu}{6(1-\nu)}a^{\alpha\beta}\Delta^2 W\right]+\frac{1}{E}\bigg(\nabla^\beta Q^\alpha+\nabla^\alpha Q^\beta\\&+\frac{2\nu}{1-\nu}a^{\alpha\beta}\nabla_\lambda Q^\lambda+\frac{\nu}{(1-\nu)h}a^{\alpha\beta}Q^3\bigg)\bigg\}\\&+\frac{E(x^3)^3}{6(1-\nu^2)}[(2-\nu)\Delta\nabla^\beta\nabla^\alpha W+\nu a^{\alpha\beta}\Delta^2 W].\end{aligned} \tag{18}$$

3. "EXTRA" INTEGRALS IN TR THEORY OF PLATE BENDING

Let us consider eqn (17). This equation differs from the classical resolving equation of the theory of plate bending:

$$\frac{2Eh^3}{3(1-\nu^2)}\Delta^2 W+2\rho h\frac{\partial^2 W}{\partial t^2}-2h\nabla_\lambda Q^\lambda-2Q^3 = 0, \tag{19}$$

only in the terms having h^2 in brackets. As could have been expected, all these terms appear if we take into account the terms of order η^{2-2q} with respect to the unity, and their role is twofold. On the one hand, they allow us to get a better approximation for the solutions of the Kirchhoff equation, and on the other hand, they are responsible for the existence of additional solutions, since the term of the equation with h^2 in the first brackets in (17) increases the order of the resolving equation of the theory of plate bending, the new order being 6 instead of 4. It is easy to see that the additional integrals have the variation index $q = 1$. This fact contradicts the first inequality (5), which is essential for the asymptotic derivation of the theory of plate bending. Thus, the increase in the order of differential equations (natural for the theories taking into account transversal shear deformation and rotation inertia) is accompanied by the appearance of some "extra" integrals mentioned in the second remark of Section 2. We can eliminate these integrals retaining the order of accuracy $[O(\eta^{4-4q}) = 0]$ as follows.

First, we apply the operator Δ to eqn (19) and find the expression for $\Delta^3 W$ from the resulting equation. We then substitute the expression found for $\Delta^3 W$ in the first brackets in the left-hand side of (17). Thus we obtain the following fourth-order resolving equation of the theory of plate bending

$$\begin{aligned}&\frac{2Eh^3}{3(1-\nu^2)}\Delta^2 W+2\rho h\left[1+h^2\frac{7\nu-17}{15(1-\nu)}\Delta\right]\frac{\partial^2 W}{\partial t^2}\\&\quad-2h\left[1-h^2\frac{4+\nu}{30(1-\nu)}\Delta\right]\nabla_\lambda Q^\lambda-2\left[1-h^2\frac{8-3\nu}{10(1-\nu)}\Delta\right]Q^3 = 0,\end{aligned} \tag{20}$$

which appears to be more correct from the mathematical point of view. One can easily verify that the order of the terms neglected in the process of its derivation is the same as for eqn (17).

In the case of static bending under transversal load ($Q^\alpha = 0$, $\partial/\partial t = 0$) eqn (20) is identical to the resolving equation found in Vijayakumar (1988), and in the case of free vibration problems ($Q^\alpha = Q^3 = 0$) it coincides with the equation established in Berdichevskii (1973) by a variational method.

One can find in Berdichevskii (1973) some references to the papers where the resolving equations take the same form after a proper correction.

Note that in the case of a free vibration problem the resolving equation of the engineering TR theories [see e.g. Ainola and Nigul (1965)] can be transformed to the form of (20), if we set the shear coefficient k^2 equal to $5/(6-\nu)$.

4. BOUNDARY CONDITIONS

Now we take a brief look at the boundary conditions in two-dimensional theories of shells and plates. The interpretations of the boundary conditions may vary according to the principles underlying particular theories. In the classical Kirchhoff–Love theory and in the engineering TR theories boundary conditions are often formulated according to the two-dimensional variational principles. The forces and the moments are assumed to perform work on the displacements of the middle surface and the rotation angles, respectively. It has been shown in Goldenveizer (1953) that the order of the error in this case is the same as that of the Kirchhoff–Love assumptions.

The boundary conditions obtained in this way are called here variational boundary conditions. Note that their number depends on the order of the corresponding differential equations. Thus, the Kirchhoff theory and the Reissner theory of plate bending require two and three boundary conditions, respectively.

Boundary conditions for the differential equations of the interior SSS are obtained in the asymptotic theories as constraints which guarantee the exponential decay of boundary layers (Goldenveizer, 1969, 1976). The order of their accuracy may vary. Thus, the so-called [see Goldenveizer (1976)] canonical boundary conditions have accuracy $[O(\eta^{1-q}) = 0]$ and coincide with the variational conditions of the classical Kirchhoff–Love theory. When the accuracy is of order $[O(\eta^{2-2q}) = 0]$, we have the so-called modified boundary conditions. In the theory of plate bending, in particular, for the free edge, the modified boundary conditions have the form:

$$N - \frac{\partial H}{\partial s} = 0, \quad G - 3Dh\frac{\partial H}{\partial s} = 0, \tag{21}$$

where N is the transversal force, G and H are, respectively, the bending and twisting moments, s is the arc length.

The value of the coefficient $3D$ is found in Kolos (1965) as a result of solving the anti-plane problem of elasticity in a semi-strip, and is given by the formula

$$3D = \frac{384}{\pi^5}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5} \approx 1.26009. \tag{22}$$

The fact that in the engineering TR theories a larger number of boundary conditions can be satisfied (e.g. three boundary conditions in the Reissner theory of plate bending) does not necessarily mean that the corresponding two-dimensional model is closer to that of the three-dimensional elasticity. The asymptotic analysis given in Kolos (1966) and Reissner (1980) shows that from the three variational boundary conditions for the interior SSS described by the Kirchhoff equation one can extract two boundary conditions of type (21), where the coefficient $3D$ is close to the value given by (22).

Thus, both methods for the solution of static boundary value problems of plate bending, i.e. "the Kirchhoff equation combined with the modified boundary conditions", on the one hand, and "the Reissner equation combined with the three variational boundary conditions", on the other hand, can be expected to produce close numerical results at large distances from the edges.

A thorough consideration of the boundary conditions and of related topics has been given in Goldenveizer (1966, 1969, 1976), Gusein-Zade (1965a, b, 1977), Koiter and Simmonds (1973), Simmonds (1976), Reissner (1980) and Gregory and Wan (1984, 1985, 1988).

5. ACCURACY AND RANGE OF APPLICABILITY OF ASYMPTOTIC THEORIES

Let us assume that the "extra" integral corresponding to the differential equations of the asymptotic TR theory has been eliminated, and therefore the order of these equations is not larger than the order of the equations in the KL theory. Thus, for instance, the resolving equation of plate bending is of order 4 and can be written in the form (20).

The above considerations show that the accuracy of the canonical boundary conditions $[O(\eta^{1-q}) = 0]$, as well as of the modified boundary conditions $[O(\eta^{2-2q}) = 0]$, is worse than the accuracy $[O(\eta^{4-4q}) = 0]$ of the differential equations of the asymptotic TR theory. Therefore, in order to refine the description of the three-dimensional SSS of a thin elastic body one has to refine not only the differential equations, but also the boundary conditions. Neither the canonical boundary conditions nor the modified ones are compatible with these requirements on the asymptotic TR theory. Therefore, it seems necessary to have a more clear picture of the effect of this discrepancy (between the accuracy of the differential equations and the accuracy of the boundary conditions) on the complete error of the solutions of static and dynamic problems of the theory of shells and plates.

As shown in Goldenveizer and Kaplunov (1988), for the static case, the solutions of the differential equations obtained with accuracy $[O(\eta^{\rho'}) = 0]$ have the error $\varepsilon' = O(\eta^{\rho'})$. Similarly, it seems natural to assume that the boundary conditions derived to within the order $[O(\eta^{\rho''}) = 0]$ also produce solution errors of the order $\varepsilon'' = O(\eta^{\rho''})$. Therefore, it is not quite consistent to employ the asymptotic TR theory for solving the static problems with the canonical or modified boundary conditions, since we always have $\rho'' < \rho'$, and, generally speaking, the increase in the accuracy of the equations will be neutralized by the errors in the boundary conditions (at least for the problems where the influence of the boundary conditions is strong enough).

In dynamic problems [see Goldenveizer and Kaplunov (1988) and Kaplunov (1990)] the differential equations derived to within the order $[O(\eta^{\rho'}) = 0]$ tend to produce larger errors in the solutions, $\varepsilon' = O(\eta^{\rho'-q})$, whereas the boundary conditions derived to within the order $[O(\eta^{\rho''}) = 0]$ cause the errors $\varepsilon'' = O(\eta^{\rho''})$ in the solutions. Thus, in the dynamic problems with sufficiently large values of the variation index q the increase in the accuracy of the differential equations of the theory of shells and plates is capable of increasing the accuracy of the solutions of the boundary value problems. This can be realized if one imposes the condition $\rho' - q < \rho''$. Then, in particular, the following inequalities are obtained:

$$q > 1/2, \quad q > 3/4, \tag{23}$$

if we use the canonical boundary conditions for the KL theory ($\rho' = 2-2q$, $\rho'' = 1-q$) and for the asymptotic TR theory ($\rho' = 4-4q$, $\rho'' = 1-q$), respectively. For both these theories the above inequalities determine the intervals of the variation index q, which guarantee that the final error for the dynamic problems depends only on the accuracy of the differential equations.

The range of applicability of any two-dimensional theory of shells and plates is determined by the inequality $q < 1$, which provides the upper bound for the admissible values of the variation index. For static problems this inequality establishes the upper bound for the applicability of both the KL theory and the asymptotic TR theory. In the case of the dynamic problems the restrictions on q become stronger: the number $\rho' - q$ must be positive (Goldenveizer and Kaplunov, 1988; Kaplunov, 1990). This leads to the inequalities:

$$q < 2/3, \quad q < 4/5, \tag{24}$$

respectively, for the KL theory and for the asymptotic TR theory. Therefore in the dynamic case the asymptotic TR theory, as compared with the KL theory, not only allows to find more precise values for the parameters of the plate's or shell's SSS (if we disregard the errors due to the formulation of the boundary conditions), but has an additional essential advantage: its range of applicability is wider. This advantage gives ground for considering vibrations of higher frequencies in stationary problems and allows a closer approach to the wave fronts in non-stationary ones.

However, there is a simpler way (than using the asymptotic TR theory) to extend the range of applicability of the KL theory to the upper limit given by the second inequality (24). In the case of plate bending one has only to introduce the terms of order $O(\eta^{2-2q})$ from the homogeneous ($Q^\alpha = Q^3 = 0$, $q = a/2+1/2$) resolving eqn (20) of the asymptotic TR theory to the homogeneous resolving equation of the Kirchhoff theory. This corresponds to the following replacement in (19):

$$2\rho h \frac{\partial^2 W}{\partial t^2} \to 2\rho h\left[1+h^2\frac{7\nu-17}{15(1-\nu)}\Delta\right]\frac{\partial^2 W}{\partial t^2}. \tag{25}$$

The homogeneous resolving equation of the Kirchhoff theory after this replacement becomes identical with the homogeneous resolving equation of the asymptotic TR theory. No more changes of the relations of the Kirchhoff theory are necessary. Thus we neglect in (11)–(14) all asymptotically small terms containing η^{2-2q}, η^{2q-2a} together with the second terms in brackets by the functions Q^3 and $\nabla_\lambda Q^\lambda$ in (20). The range of applicability of the KL theory taking into account the replacement (25) will be defined by the second inequality (24), which corresponds to the range of applicability of the asymptotic TR theory.

Let us briefly discuss the approach suggested above. To this end we point out the source of error reducing the range of applicability of the KL theory in dynamic problems. It is shown in Goldenveizer and Kaplunov (1988) and Kaplunov (1990) for the case of shells that the first restriction (24) is linked to the error of finding propagating vibration modes (i.e. particular solutions of the homogeneous equations of motion, which correspond to the SSS extended over the entire body). A similar situation naturally arises in bending of plates, which can be illustrated by an example of the following one-dimensional stationary problem in a strip of length R. Set $\partial/\partial t = -i\omega$, $Q^\alpha = Q^3 = \partial/\partial x^2 = 0$ in (19), keeping in mind that to obtain this relation the terms of order $O(\eta^{2-2q})$ with respect to unity were neglected. Then the propagating modes with $q < 1$ have the form ($q = 1/2+a/2$):

$$\exp(\pm i\eta^{-q}\sqrt{\omega_1}[1+O(\eta^{2-2q})]x^1/R), \tag{26}$$

where

$$\omega_1 = \eta^a\frac{\omega R}{c_{sh}}\sqrt{3(1-\nu^2)} \quad (\omega_1 \sim \eta^0).$$

Let us require that the modes (26) can be found from eqn (19) over the whole length of the strip, i.e. when $x^1/R \sim \eta^0$. Then (26) directly implies that the O-term in the exponent can be neglected only if the first inequality (24) is satisfied.

The error in the calculation of the propagating vibration modes is the only reason for the reduction of the range of applicability of the Kirchhoff theory in dynamic problems. Obviously, the error of order $O(\eta^{2-2q})$ appears after neglecting the asymptotically secondary terms in the two last brackets in the resolving equation (20) and the terms containing η^{2-2q}, η^{2q-2a} in (11)–(14), when the three-dimensional SSS of the plate is being reconstructed on the basis of the given W.

The above considerations seem to provide a rational explanation for the fact that in dynamic problems the engineering TR theories, which are developed on the basis of physical assumptions, in spite of being asymptotically inconsistent,† often allow better results to be obtained than the Kirchhoff theory. In this relation one should note that all values of the shear modulus k^2 suggested in the engineering TR theories [see e.g. Ainola and Nigul (1965) and Grigolyuk and Selezov (1973)] are numerically close to the value $k^2 = 5/(6-\nu)$, which, as one can easily verify, allows a reduction (to within the asymptotically secondary terms) of the resolving equation of the engineering TR theories to the form (19), if one takes into account the replacement (25). Therefore the range of applicability of the engineering TR theories turns out to be wider than that of the Kirchhoff theory.

6. ASYMPTOTIC TR THEORY OF PLATE EXTENSION AND TRANSVERSAL COMPRESSION

Consider now the problem of extension and transversal compression of a plate. We shall construct in this case the asymptotic theories of type $[O(\eta^{2-2q}) = 0]$ and of type $[O(\eta^{4-4q}) = 0]$ and regard them as a KL theory and a TR theory, respectively, for the SSS of this form.

We take the following face conditions:

$$\sigma^{33}|_{x^3=\pm h} = Q^3, \quad \sigma^{3\alpha}|_{x^3=\pm h} = \pm Q^\alpha, \tag{27}$$

where $Q^3 = 0$ corresponds to the extension problem and $Q^\alpha = 0$ corresponds to the problem of transversal compression. Since we are going to study vibrations with predominantly tangential displacements, let us assume that the second inequality (6) holds. In addition, we still assume that the indices q and a satisfy the inequalities (5).

The theory of extension and transversal compression is derived in the same way as the theory of bending. Therefore the comments will be reduced to a minimum. We specify the asymptotics of the SSS by the relations:

$$u_\alpha = Ru_\alpha^*, \quad w = R\eta^{1-q}w^*, \quad \sigma^{\alpha\beta} = E\eta^{-q}\sigma_*^{\alpha\beta}, \quad \sigma^{3\alpha} = E\eta^{1-2q}\sigma_*^{3\alpha}, \quad \sigma^{33} = E\eta^{-q}\sigma_*^{33}, \tag{28}$$

which, in particular, imply that by virtue of the first inequality (5) among the displacement and stress components u_α and $\sigma^{\alpha\beta}$, σ^{33}, respectively, are asymptotically principal, whereas w, $\sigma^{3\alpha}$ are asymptotically secondary.

Remark. Formulas (28) determine the asymptotic properties of a plate simultaneously subjected to extension and transversal compression. Here the intensities of the tangential and normal loads are assumed such that the asymptotically principal parameters u_α and $\sigma^{\alpha\beta}$ of the plate's SSS produced by extension have the same order as the above parameters of the SSS produced by transversal compression. This is the case when Q^α and $\eta^{1-q}Q^3$ have the same order. If $|Q^\alpha| \ll \eta^{1-q}|Q^3|$, or $|Q^\alpha| \gg \eta^{1-q}|Q^3|$ transversal compression or, respectively, extension will predominate. In the first case the asymptotic order of $\sigma^{3\alpha}$ should be decreased (in the limit one has: $Q^\alpha = 0$, $\sigma^{3\beta} = E\eta^{3-4q}\sigma_*^{3\beta}$). In the second case the asymptotic order of σ^{33} should be decreased (in the limit one has $Q^3 = 0$, $\sigma^{33} = E\eta^{2-3q}\sigma_*^{33}$, i.e. the stress component σ^{33} becomes asymptotically secondary).

Using the transformation of the independent variables (4) and taking into account (28) we rewrite the equations of three-dimensional elasticity (8), (9) in the form

$$\frac{\partial \sigma_*^{3\beta}}{\partial \zeta} = -\tilde{\nabla}_\alpha \sigma_*^{\alpha\beta} + \eta^{2q-2a}\frac{\partial^2 u_*^\beta}{\partial \tau^2},$$

$$\frac{\partial \sigma_*^{33}}{\partial \zeta} = -\eta^{2-2q}\tilde{\nabla}_\alpha \sigma_*^{3\alpha} + \eta^{2-2a}\frac{\partial^2 w^*}{\partial \tau^2},$$

$$\frac{\partial w^*}{\partial \zeta} = \sigma_*^{33} - \nu a_{\lambda\mu}\sigma_*^{\lambda\mu},$$

† The asymptotic inconsistency of the engineering TR theories is discussed in Section 7.

$$\frac{\partial u_\alpha^*}{\partial \zeta} = -\eta^{2-2q}\tilde{\nabla}_\alpha w^* + 2(1+\nu)\eta^{2-2q}a_{\lambda\alpha}\sigma_*^{3\lambda},$$

$$\sigma_*^{\alpha\beta} = \frac{1}{1-\nu^2}[(1-\nu)e_*^{\alpha\beta} + \nu a^{\alpha\beta}a_{\lambda\mu}e_*^{\lambda\mu}] + \frac{\nu}{1-\nu}a^{\alpha\beta}\sigma_*^{33},$$

$$e_{\alpha\beta}^* = \tfrac{1}{2}(\tilde{\nabla}_\beta u_\alpha^* + \tilde{\nabla}_\alpha u_\beta^*). \tag{29}$$

In these equations all exponents of η are non-negative by virtue of (5) and the second inequality (6). When $q > a$ all inertial terms in (29) are asymptotically secondary and these equations correspond to the quasi-static case (or to the static one as $a \to -\infty$). The free tangential vibrations of the plate correspond to the case $a = q$.

It can be verified that to within the order $[O(\eta^{4-4q}) = 0]$ the system (29) possesses solutions such that the functions marked by the asterisk in (28) are given by the formulas:

$$w^* = \zeta w^{(1)} + \zeta^3\eta^{2-2q}w^{(3)},$$

$$u_\alpha^* = u_\alpha^{(0)} + \zeta^2\eta^{2-2q}u_\alpha^{(2)},$$

$$e_*^{\alpha\beta} = e_{(0)}^{\alpha\beta} + \zeta^2\eta^{2-2q}e_{(2)}^{\alpha\beta},$$

$$\sigma_*^{\alpha\beta} = \sigma_{(0)}^{\alpha\beta} + \zeta^2\eta^{2-2q}\sigma_{(2)}^{\alpha\beta},$$

$$\sigma_*^{3\beta} = \zeta\sigma_{(1)}^{3\beta} + \zeta^3\eta^{2-2q}\sigma_{(3)}^{3\beta},$$

$$\sigma_*^{33} = \sigma_{(0)}^{33} + \zeta^2\eta^{2-2q}\sigma_{(2)}^{33}. \tag{30}$$

Formulas (30) describe the variation of the SSS with respect to the transversal variable ζ. Functions in (30) with additional numerical indices in parentheses depend only on ξ_1, ξ_2, τ and are related by the formulas:

$$e_{\alpha\beta}^{(0)} = \tfrac{1}{2}(\tilde{\nabla}_\beta u_\alpha^{(0)} + \tilde{\nabla}_\alpha u_\beta^{(0)}),$$

$$\sigma_{(0)}^{33} = Q_*^3 - \eta^{2-2q}\sigma_{(2)}^{33},$$

$$\sigma_{(0)}^{\alpha\beta} = \frac{1}{1-\nu^2}[(1-\nu)e_{(0)}^{\alpha\beta} + \nu a^{\alpha\beta}a_{\lambda\mu}e_{(0)}^{\lambda\mu}] + \frac{\nu}{1-\nu}a^{\alpha\beta}\sigma_{(0)}^{33},$$

$$\sigma_{(1)}^{3\beta} = -\tilde{\nabla}_\alpha\sigma_{(0)}^{\alpha\beta} + \eta^{2q-2a}\frac{\partial^2 u_{(0)}^\beta}{\partial\tau^2}, \tag{31}$$

and

$$\sigma_{(1)}^{3\beta} + \eta^{2-2q}\sigma_{(3)}^{3\beta} = Q_*^\beta, \tag{32}$$

and

$$w^{(1)} = -\nu a_{\lambda\mu}\sigma_{(0)}^{\lambda\mu} + \sigma_{(0)}^{33},$$

$$u_\alpha^{(2)} = -\tfrac{1}{2}\tilde{\nabla}_\alpha w^{(1)} + (1+\nu)a_{\lambda\alpha}\sigma_{(1)}^{3\lambda},$$

$$e_{\alpha\beta}^{(2)} = \tfrac{1}{2}(\tilde{\nabla}_\beta u_\alpha^{(2)} + \tilde{\nabla}_\alpha u_\beta^{(2)}),$$

$$\sigma_{(2)}^{33} = -\tfrac{1}{2}\tilde{\nabla}_\alpha\sigma_{(1)}^{3\alpha} + \tfrac{1}{2}\eta^{2q-2a}\frac{\partial^2 w^{(1)}}{\partial\tau^2},$$

$$\sigma_{(2)}^{\alpha\beta} = \frac{1}{1-\nu^2}[(1-\nu)e_{(2)}^{\alpha\beta} + \nu a^{\alpha\beta}a_{\lambda\mu}e_{(2)}^{\lambda\mu}] + \frac{\nu}{1-\nu}a^{\alpha\beta}\sigma_{(2)}^{33},$$

$$\sigma_{(3)}^{3\beta} = -\tfrac{1}{3}\tilde{\nabla}_\alpha\sigma_{(2)}^{\alpha\beta} + \tfrac{1}{3}\eta^{2q-2a}\frac{\partial^2 u_{(2)}^\beta}{\partial\tau^2}, \tag{33}$$

and

$$w^{(3)} = -\tfrac{1}{3}\nu a_{\lambda\mu}\sigma_{(2)}^{\lambda\mu} + \tfrac{1}{3}\sigma_{(2)}^{33}, \tag{34}$$

where

$$Q_*^\beta = E^{-1}\eta^{2q-1}Q^\beta, \quad Q_*^3 = E^{-1}\eta^q Q^3.$$

The system (31)–(34) allows two-dimensional equations of the theory of extension and transversal compression of plates to be derived to within the order $[O(\eta^{2-2q}) = 0]$, which corresponds to the KL theory, and also to within the order $[O(\eta^{4-4q}) = 0]$, which corresponds to the asymptotic TR theory.

In the first case we should neglect in (31)–(34) the terms containing η^{2-2q}, so that the equalities (31) and (32) become a complete system of equations with respect to the unknowns $u_\alpha^{(0)}$ and

$$e_{\alpha\beta}^{(0)}, \sigma_{(0)}^{\alpha\beta}, \sigma_{(1)}^{3\beta}, \sigma_{(0)}^{33}. \tag{35}$$

This system corresponds to the equations of the KL theory of extension and transversal compression of plates, which determines the asymptotically principal parameters of the plate's SSS. Let us consecutively express the quantities (35) in terms of $u_\alpha^{(0)}$ and Q^3, using the equalities obtained from (31) after neglecting the terms containing η^{2-2q}, and then substitute $\sigma_{(1)}^{3\beta}$ into the simplified equality (32). This leads to the following system of differential equations (written in the initial independent variables x^α, t) for $u_\alpha^{(0)}$:

$$Eh\left[\frac{1}{1+\nu}\Delta U^\beta + \frac{1}{1-\nu}\nabla^\beta\nabla_\lambda U^\lambda\right] - 2\rho h\frac{\partial^2}{\partial t^2}U^\beta + 2Q^\beta + \frac{2h\nu}{1-\nu}\nabla^\beta Q^3 = 0. \tag{36}$$

Here $U^\alpha = u^\alpha|_{x^3=0} = Ru_{(0)}^\alpha$. Further, assuming $u_\alpha^{(0)}$ to be known and using the corresponding formulas in (30) written to within $[O(\eta^{2-2q}) = 0]$, we express the asymptotically principal parameters of the plate's SSS u_α, $e_{\alpha\beta}$, $\sigma^{\alpha\beta}$, σ^{33} in terms of the functions (35) and, therefore, the functions $u_\alpha^{(0)}$.

When $Q^3 = 0$, eqns (36) coincide with the equations of the classical plane elasticity determining the generalized plane stress state.

In order to construct the asymptotic theory of type $[O(\eta^{4-4q}) = 0]$ for the extension and transversal compression of plates let us first express the quantities (35) to within the order $[O(\eta^{2-2q}) = 0]$ in terms of $u_\alpha^{(0)}$ and Q^3, using the simplified formulas (31), as described above. On the basis of (33) we then succesively find expressions for

$$w^{(1)}, u_\alpha^{(2)}, e_{\alpha\beta}^{(2)}, \sigma_{(2)}^{33}, \sigma_{(2)}^{\alpha\beta}, \sigma_{(3)}^{3\beta} \tag{37}$$

in terms of $u_\alpha^{(0)}$ and Q^3. Now we construct corrections of the order η^{2-2q} for the quantities (35), defined by (31). To this end we replace in (31) the previously neglected term with $\sigma_{(2)}^{33}$ by its approximation. The equations for $u_\beta^{(0)}$ to within the order $[O(\eta^{4-4q}) = 0]$ are obtained after substituting in the unsimplified equality (32) $\sigma_{(1)}^{3\beta}$ and $\sigma_{(3)}^{3\beta}$ expressed through $u_\beta^{(0)}$ and Q^3 to within the order $[O(\eta^{4-4q}) = 0]$ and $[O(\eta^{2-2q}) = 0]$, respectively. In the initial independent variables these equations take the form

$$\begin{aligned}
&Eh\left(\frac{1}{1+\nu}\Delta U^\beta + \frac{1}{1-\nu}\nabla^\beta\nabla_\lambda U^\lambda\right) - 2\rho h\frac{\partial^2}{\partial t^2}U^\beta + 2Q^\beta + \frac{2h\nu}{1-\nu}\nabla^\beta Q^3 \\
&+ Eh^3\left[-\frac{1}{6(1+\nu)}\Delta^2 U^\beta - \frac{3+4\nu-\nu^2}{6(1-\nu)^2(1+\nu)}\Delta\nabla^\beta\nabla_\lambda U^\lambda\right. \\
&+ \frac{2\rho}{3E}\frac{\partial^2}{\partial t^2}\Delta U^\beta - \frac{2(1+\nu)\rho^2}{3E^2}\frac{\partial^4}{\partial t^4}U^\beta + \frac{(2+\nu-\nu^2)\rho}{3(1-\nu)^2 E}\frac{\partial^2}{\partial t^2}\nabla^\beta\nabla_\lambda U^\lambda \\
&\left.- \frac{1+2\nu^2}{3(1-\nu)^2 E}\Delta\nabla^\beta Q^3 + \frac{(1+\nu)(1-3\nu+4\nu^2)\rho}{3(1-\nu)^2 E^2}\frac{\partial^2}{\partial t^2}\nabla^\beta Q^3\right] = 0.
\end{aligned} \tag{38}$$

Assuming the quantities (35), (37) are known (i.e. calculated to within the order $[O(\eta^{4-4q}) = 0]$ and $[O(\eta^{2-2q}) = 0]$, respectively), we can apply formulas (28), (30) to construct, in the first place, the asymptotically secondary parameters of the plate's SSS w and $\sigma^{3\beta}$ to within the order $[O(\eta^{2-2q}) = 0]$ [to this end we should neglect the terms with η^{2-2q} in the corresponding formulas (30)], and, secondly, to refine the asymptotically principal SSS's parameters, i.e. to find u_α, $e_{\alpha\beta}$, σ^{33} to within the order $[O(\eta^{4-4q}) = 0]$. This amounts to the construction of the asymptotic TR theory for the extension and transversal compression of plates.

Note that the formulas (30)–(34) also allow the asymptotically secondary parameters of the plate's SSS to be found to within the order $[O(\eta^{4-4q}) = 0]$. For instance, to find the transversal displacement w to within the order $[O(\eta^{4-4q}) = 0]$, we should first find $w^{(3)}$ from (34) to within the order $[O(\eta^{2-2q}) = 0]$, and then by another iteration to find $w^{(1)}$ from (33) to within the order $[O(\eta^{4-4q}) = 0]$.

Consider again the system (38). In analogy to the problem of bending, let us eliminate the "extra" integrals, which amounts to reducing the order (in x^α) of the equations. The system (38) differs from (36) only by the terms in brackets which appear if we take into account terms of order $O(\eta^{2-2q})$ with respect to unity. Consecutively applying to (36) the operators $\nabla^\alpha\nabla_\beta$, Δ and $\partial^2/\partial t^2$ we, respectively, obtain three equalities, which, after simple transformations, yield the expressions for $\Delta\nabla^\alpha\nabla_\lambda U^\lambda$, $\Delta^2 U^\alpha$ and $\partial^4 U^\alpha/\partial t^4$ to within the order $[O(\eta^{2-2q}) = 0]$. Substituting these functions in (38) we get the following equations:

$$\begin{aligned} &Eh\left(\frac{1}{1+\nu}\Delta U^\beta + \frac{1}{1-\nu}\nabla^\beta\nabla_\lambda U^\lambda\right) - 2\rho h\frac{\partial^2}{\partial t^2}U^\beta + 2Q^\beta + \frac{2h\nu}{1-\nu}\nabla^\beta Q^3 \\ &+ \frac{2\nu^2}{3(1-\nu)^2}\rho h^3\frac{\partial^2}{\partial t^2}\nabla^\beta\nabla_\lambda U^\lambda + \tfrac{1}{3}h^2\Delta Q^\beta + \frac{1+2\nu}{3(1-\nu)}h^2\nabla^\beta\nabla_\lambda Q^\lambda \\ &- \frac{2(1+\nu)\rho}{3E}h^2\frac{\partial^2}{\partial t^2}Q^\beta - \tfrac{1}{3}h^3\Delta\nabla^\beta Q^3 \\ &+ \frac{(1+\nu)(1-3\nu)(1-2\nu)\rho}{3(1-\nu)^2 E}h^3\frac{\partial^2}{\partial t^2}\nabla^\beta Q^3 = 0, \end{aligned} \tag{39}$$

whose accuracy is the same as above, i.e. $[O(\eta^{4-4q}) = 0]$.

The range of applicability of eqns (39) in dynamic problems is given by the inequality $a = q < 4/5$ according to (24) and the second inequality (6).

One of the important advantages of the system (39) is that in nonstationary dynamics it allows us to obtain an adequate description of the three-dimensional solution in a neighborhood of the quasi-front (the front of the longitudinal wave in the classical problem of plane elasticity) [see Kaplunov and Nolde (1992a)].

If the primary aim in dynamic problems were to expand the range of applicability of the equations of the KL theory, then, similarly to the problem of plate bending, it would suffice to obtain with greater accuracy the homogeneous resolving system of equations. This can be done by means of the following replacement:

$$2\rho h\frac{\partial^2}{\partial t^2}U^\beta \to 2\rho h\frac{\partial^2}{\partial t^2}\left[U^\beta - \frac{\nu^2}{3(1-\nu)^2}h^2\nabla^\beta\nabla_\lambda U^\lambda\right], \tag{40}$$

in the homogeneous resolving system of the classical plane elasticity (36) ($Q^\alpha = Q^3 = 0$).

7. PHYSICAL ASSUMPTIONS CORRESPONDING TO ASYMPTOTIC THEORIES OF PLATES

Let us go back to the relations (7)–(9), (11)–(14), and (28)–(34) obtained for the theory of plate bending and the theory of plate extension and transversal compression, respectively.

These equalities establish the following properties in accordance with the exponent of the factor η^p:

(a) the asymptotic form of the corresponding SSS [see (7) and (28)];

(b) the asymptotic order of the separate terms in the equations of three-dimensional elasticity [see (8), (9) and (29)];

(c) the dependence on the transversal variable of the functions describing the given SSS [see (11) and (30)].

The remaining equalities (12)–(14) and (31)–(34) allow the construction of the two-dimensional equations of the theory of plate bending and of the theory of plate extension and transversal compression.

On the basis of these considerations we can formulate physical KL and TR assumptions, which reflect the choice of neglected terms in the process of derivation of the equations and other formulas in the KL theory and the asymptotic TR theory, respectively.

In the theory of plate bending we are led to the following assumptions by virtue of the relations (7)–(9), (11)–(14).

KL assumption 1. The transversal shear deformation is negligibly small.

KL assumption 2. The Poisson influence of the stress component σ^{33} on $\sigma^{\alpha\beta}$ is negligibly small.

KL assumption 3. The functions determining the plate's SSS are represented by the following geometric figures, symmetrical or antisymmetrical with respect to the midplane: a symmetrical straight line (for w), an antisymmetrical straight line (for u_α, $\sigma^{\alpha\beta}$), a symmetrical parabola of second order (for $\sigma^{3\beta}$), an antisymmetrical parabola of third order (for σ^{33}).

KL assumption 4. The tangential forces of inertia are negligibly small.

Likewise we obtain the following assumptions for the asymptotic TR theory.

TR assumption 1. The variation of the length of the normal element should be taken into account as a KL quantity related merely to the Poisson effect.

We say that a certain property *is taken into account as a KL quantity* if the parameters or functions determining that property are calculated on the basis of the KL assumptions rather than the TR assumptions.

In order to clarify the meaning of the expression "to take into account as a KL quantity", consider the TR assumption 1.

According to the KL assumptions, the first equation (2) has the form

$$\frac{\partial w}{\partial x^3} = 0.$$

According to the TR assumption 1 this equation should be written in the form

$$\frac{\partial w}{\partial x^3} = -\nu a_{\lambda\mu}\sigma^{\lambda\mu}.$$

Here the functions $\sigma^{\lambda\mu}$ are calculated on the basis of the KL theory (all KL assumptions are taken into account; in particular, the dependence of $\sigma^{\lambda\mu}$ on x^3 is linear).

TR assumption 2. The transversal shear deformation should be taken into account only as a KL quantity.

TR assumption 3. The Poisson influence of the stress component σ^{33} on the components $\sigma^{\alpha\beta}$ should be taken into account only as a KL quantity.

TR assumption 4. The unknown functions determining the plate's SSS are represented by the following geometric figures: a symmetrical parabola of second order (for w), an antisymmetrical parabola of third order (for u^α, $\sigma^{\alpha\beta}$), a symmetrical parabola of fourth order (for $\sigma^{3\beta}$), an antisymmetrical parabola of fifth order (for σ^{33}).

TR assumption 5. Tangential inertia forces should be taken into account as a KL quantity and, therefore, the rotation inertia must be taken into account.

In general, the KL assumptions are equivalent to the usual Kirchhoff–Love assumptions. The assumptions underlying the engineering TR theories are not completely identical to the above TR assumptions. This fact accounts for the asymptotical inconsistency of the engineering TR theories.

A similar conclusion about the asymptotical inconsistency of the equations of the engineering TR theories in the case of the general static theory of shells is arrived at in Rogacheva (1974).

It should be stressed once again that as a rule, in spite of being asymptotically inconsistent, the engineering TR theories, applied to dynamic problems, actually have a wider range of applicability than the KL theories (see Section 5).

Let us consider now to what extent one is justified in introducing in the KL theory and the asymptotic TR theory such two-dimensional quantities as forces and moments, as well as displacements and rotation angles of the midplane. The KL assumption 3 establishes a linear antisymmetrical dependence on the thickness parameter for stress components $\sigma^{\alpha\beta}$ which are asymptotically principal in the theory of plate bending. This gives rational ground for introducing the notion of moments in the KL theory, since one can obviously reconstruct the stress components with no further errors, provided that the moments are known. The TR assumption 4 implies that $\sigma^{\alpha\beta}$ varies according to the law of third-order parabola [see also (11)]. This fact seems to be unfavorable for using the notion of moments in the asymptotic TR theory of plate bending, since additional efforts are needed in order to find stresses to within the order $[O(\eta^{4-4q}) = 0]$, provided that the moments are known. Likewise in the framework of the asymptotic TR theory the knowledge of the transversal displacement W of the midplane is not sufficient for finding to within the given order the more important quantities such as the transversal displacements of the faces.

The KL and TR assumptions for the theory of plate extension and transversal compression will be formulated here only in the case of extension [in (27) $Q^3 = 0$].

Let us assume that the problem of plate extension in the KL theory consists in finding only the asymptotically principal quantities u_α, $\sigma^{\alpha\beta}$ and also the component $\sigma^{3\beta}$. Then, taking into account (28)–(34) and the remark in Section 6, we can state that:

The KL assumption 1 is the same as the KL assumption 1 in the theory of bending.

KL assumption 2. The functions determining the plate's SSS are represented by a symmetrical straight line (for u_α, $\sigma^{\alpha\beta}$) and an antisymmetrical straight line (for $\sigma^{3\beta}$).

These assumptions allow a complete subsystem with respect to the unknowns u_α, $\sigma^{\alpha\beta}$ and $\sigma^{3\beta}$ together with tangential face conditions to be singled out from among the equations of three-dimensional elasticity. This system has the form

$$\begin{aligned}
\nabla_\alpha \sigma^{\alpha\beta} + \frac{\partial \sigma^{3\beta}}{\partial x^3} - \rho \frac{\partial^2 u^\beta}{\partial t^2} &= 0, \\
\frac{\partial u_\alpha}{\partial x^3} &= 0, \\
E e_{\alpha\beta} &= (1+\nu)\sigma_{\alpha\beta} - \nu a_{\alpha\beta} a^{\lambda\mu} \sigma_{\lambda\mu}, \\
e_{\alpha\beta} &= \tfrac{1}{2}(\nabla_\beta u_\alpha + \nabla_\alpha u_\beta), \\
\sigma^{3\beta}|_{x^3 = \pm h} &= \pm Q^\beta.
\end{aligned} \tag{41}$$

The construction of the entire SSS to within the order $[O(\eta^{2-2q}) = 0]$ requires a more elaborate formulation of the KL assumptions. However we shall not discuss this question here, but shall directly go over to the asymptotic TR theory of plate extension. According to our conception of this theory, the quantities u_α, $\sigma^{\alpha\beta}$ and $\sigma^{3\beta}$ are the unknowns to be found to within the order $[O(\eta^{4-4q}) = 0]$ and w, σ^{33} are to be found to within the order $[O(\eta^{2-2q}) = 0]$. Then, by virtue of (28)–(34), we can state that:

The TR assumptions 1 and 2 are formulated similarly to the theory of bending.

TR assumption 3. In the third equation of equilibrium $(\nabla_\alpha \sigma^{3\alpha} + \partial \sigma^{33}/\partial x^3 - \rho\, \partial^2 w/\partial t^2 = 0)$ the influence of the transversal stress components $\sigma^{3\alpha}$ should be taken into account as a KL quantity.

TR assumption 4. The functions describing the unknown SSS are represented by: a symmetrical second order parabola (for u_α, $\sigma^{\alpha\beta}$ and σ^{33}), an antisymmetrical third order parabola (for $\sigma^{3\beta}$), an antisymmetrical straight line (for w).

8. REFINED DYNAMIC THEORY OF SHELLS

Let us try to extend the results obtained for the theory of plate bending and for the plane elasticity to the dynamic theory of shells. First of all, it should be pointed out that we consider here only the extension of the range of applicability of the KL theory, defined by the first inequality (24) [to that of the asymptotic TR theory, defined by the second inequality (24)].

It is well known [see e.g. Goldenveizer (1987), Goldenveizer and Kaplunov (1988) and Kaplunov (1990)] that in a certain sense the influence of the geometrical properties of a shell on its SSS becomes secondary with the increase of the vibration frequency in stationary dynamics, as well as in a neighborhood of the wave fronts in the non-stationary dynamics (i.e. with the increase of the dynamicity index a and the variation index q). According to the above-mentioned papers, for a shell with free faces the description of the vibrations with the variation index $q > 0$ is based on the possibility of dividing (with an asymptotically negligible error) the initial system of equations of the dynamic KL theory into two simpler systems. The first one is the system of the plane theory of shells which describes quasi-tangential vibrations of the shell ($q = a$) and coincides with the system of the plane elasticity related to the metric tensor of the middle surface of the shell. The second one is the system of the bending-plane theory which describes the quasi-transversal vibrations ($q = 1/2 + a/2$). Its asymptotically principal part coincides with the equations of the theory of plate bending. The necessity of retaining the secondary plane terms of this system is established in Kaplunov (1990).

Using the above considerations, we can show that in order to extend the range of applicability of the KL theory, it suffices to perform the replacements (25) and (40) in the corresponding equations, i.e. to correct the tangential inertia terms in the same way as was done for the plane elasticity, and to correct the normal inertia terms, as in the theory of plate bending. Here we can apply all relations of the KL theory and the canonical boundary conditions which hold in the interval $q < 1$.

9. DYNAMIC THEORIES OF PLATE BENDING WITH WIDER RANGES OF APPLICABILITY

In conclusion we consider a method to derive dynamic theories of plate bending whose range of applicability is wider than that given by the second inequality (24). It should be pointed out from the start that we are concerned here only with the extension of the range of applicability and not with the construction of the three-dimensional SSS of a plate to within the order higher than $[O(\eta^{2-2q}) = 0]$. The considerations of Section 5 show that our aim can be achieved if we write out the homogeneous resolving equation of the theory of plate bending to within the order higher than $[O(\eta^{4-4q}) = 0]$. To be more precise we restrict ourselves with the accuracy $[O(\eta^{8-8q}) = 0]$.

Consider once again the system (7)–(9) with $Q^\alpha = Q^3 = 0$ in (3). Let us apply the method described in Sections 2 and 3 setting $q = (a+1)/2$ and writing out the series in powers of ζ to within the order $[O(\eta^{8-8q}) = 0]$ for all terms marked by the asterisk in (11). After lengthy but essentially simple calculations we obtain the following homogeneous resolving equation:

$$\frac{2Eh^3}{3(1-\nu^2)}\Delta^2 W + 2\rho h\left[1 + h^2 b_0 \Delta + a_1 \frac{h^2}{c_2^2}\frac{\partial^2}{\partial t^2} + b_1 \frac{h^4}{c_2^2}\frac{\partial^2}{\partial t^2}\Delta\right]\frac{\partial^2 W}{\partial t^2} = 0, \tag{42}$$

$$b_0 = \frac{7\nu - 17}{15(1-\nu)}, \quad a_1 = \frac{422 - 424\nu - 33\nu^2}{1050(1-\nu)},$$

$$b_1 = \frac{32-96\nu+261\nu^2-197\nu^3}{15750(1-\nu)^2}, \quad c_2 = \sqrt{\frac{E}{2(1+\nu)\rho}}.$$

Notice that the homogeneous resolving equation of the Kirchhoff theory of plate bending coincides with eqn (42), if we perform the replacement:

$$2\rho h \frac{\partial^2 W}{\partial t^2} \rightarrow 2\rho h\left[1+h^2 b_0 \Delta + a_1 \frac{h^2}{c_2^2}\frac{\partial^2}{\partial t^2} + b_1 \frac{h^4}{c_2^2}\frac{\partial^2}{\partial t^2}\Delta\right]\frac{\partial^2 W}{\partial t^2}. \tag{43}$$

The arguments of Section 5 show that the dynamic theory of plate bending, whose difference from the Kirchhoff theory consists only of the replacement (43) in the resolving equation (19), is valid for $q < 8/9$, $a < 7/9$ [see also Goldenveizer and Kaplunov (1988) and Kaplunov (1990)].

In the same way we can also obtain a more accurate formulation of the plane elasticity and of the theory of shells. However, we cannot here consider these equations in detail.

In the case of propagating stationary vibration modes, which develop in time as $\exp(-i\omega t)$, eqn (42) can be reduced to a more simple form. Indeed, let $\partial/\partial t = -i\omega$. Then, by means of the iteration process we can obtain from (42) the following representation for the propagating vibration modes:

$$\Delta W = -\frac{\omega}{hc_2}\sqrt{\tfrac{3}{2}(1-\nu)}\left[1+B_1\frac{\omega h}{c_2}+B_2\left(\frac{\omega h}{c_2}\right)^2+O(\eta^{6-6q})\right]. \tag{44}$$

Here B_1 and B_2 are the functions of Poisson's ratio, ν. Omitting the explicit formulas for B_1 and B_2, we directly write out the result of substituting (44) into the asymptotically secondary terms of eqn (42). To within the order $[O(\eta^{(2-2q)(n+1)}) = 0]$ $(n = 0, 1, 2, 3)$ we, thus, get

$$\frac{2Eh^3}{3(1-\nu^2)}\Delta^2 W - 2\rho h\omega_r^2 W = 0, \tag{45}$$

$$\omega_r^2 = \omega^2 \sum_{k=0}^{n} A_k\left(\frac{\omega h}{c_2}\right)^k, \tag{46}$$

where

$$A_0 = 1, \quad A_1 = \sqrt{\tfrac{3}{2}(1-\nu)}\frac{17-7\nu}{15(1-\nu)}, \quad A_2 = \frac{1179-818\nu+409\nu^2}{2100(1-\nu)},$$

$$A_3 = \sqrt{\tfrac{3}{2}(1-\nu)}\frac{5951-2603\nu+9953\nu^2-4901\nu^3}{126000(1-\nu)^2}.$$

Here n is the order of accuracy.

Equation (45) can be used, in particular, to find more accurate values for the natural frequencies than those calculated according to the Kirchhoff theory and denoted here by ω_0. Suppose that eqn (45) is supplied with the canonical boundary conditions. Assuming ω_0 to be known and taking into account the condition $\omega_r^2 = \omega_0^2$, we obtain the following equation, which allows us to find more accurate values for the natural frequencies ω_n (here n is the order of accuracy, $n = 1, 2, 3$):

Table 1. Comparison of natural frequencies, $\omega_* = 2\omega h/c_2$, of a simply-supported square plate

M	N	Exact	KL	Refined values (47)			HSDPT
				$n = 1$	$n = 2$	$n = 3$	
1	1	0.0932	0.0963	0.0932	0.0931	0.0931	0.0931
1	2	0.2226	0.2408	0.2233	0.2226	0.2226	0.2222
2	2	0.3421	0.3853	0.3446	0.3422	0.3421	0.3411
1	3	0.4171	0.4816	0.4214	0.4173	0.4171	0.4158
2	3	0.5239	0.6261	0.5317	0.5242	0.5239	0.5221
1	4	—	0.8187	0.6712	0.6577	0.6571	0.6545
3	3	0.6889	0.8669	0.7049	0.6897	0.6890	0.6862
2	4	0.7511	0.9632	0.7710	0.7521	0.7511	0.7481
3	4	—	1.2040	0.9300	0.9003	0.8985	0.8949
1	5	0.9268	1.2521	0.9607	0.9287	0.9268	0.9230
2	5	—	1.3966	1.0514	1.0119	1.0093	1.0053
4	4	1.0889	1.5411	1.1397	1.0922	1.0889	1.0847
3	5	—	1.6374	1.1972	1.1442	1.1403	1.1361

Table 2. Comparison of natural frequencies, $\omega_{**} = 2\omega h/c_1 (c_1 = [(2-2\nu)/(1-2\nu)]^{1/2}c_2)$, of a circular plate with a free edge

η	Exact	KL	Refined values		
			$n = 1$	$n = 2$	$n = 3$
0.05	0.23–0.24	0.24	0.24	0.24	0.24
	0.9–1.0	1.00	0.94	0.94	0.94
	2.0–2.1	2.29	2.03	2.01	2.01
	3.3–3.4	4.09	3.39	3.32	3.32
0.1	0.44–0.45	0.47	0.44	0.44	0.44
	1.6–1.7	2.01	1.67	1.64	1.64
	3.1–3.2	4.58	3.33	3.16	3.16
	4.7–4.8	8.19	5.25	4.84	4.80
0.2	0.78–0.79	0.95	0.79	0.78	0.78
	2.3–2.4	4.02	2.59	2.39	2.37
	3.9–4.0	9.16	4.81	4.16	4.07

$$\omega_n^2 \sum_{k=0}^{n} A_k \left(\frac{\omega_n h}{c_2}\right)^k = \omega_0^2. \tag{47}$$

To illustrate the possibilities provided by the refined equation (45) consider the problems of finding frequencies of natural vibrations of a square simply-supported plate whose side length is $b = 20h$ and of a circular plate of radius R with a free edge ($\nu = 0.3$). The corresponding numerical results are given in Tables 1 and 2. The values obtained on the basis of the Kirchhoff theory are borrowed from Reddy (1984) and Grinchenko and Komissarova (1974). Substitution of them in (47) ($n = 1, 2, 3$) gives three more accurate values ω_n. They are compared with the exact solutions found on the basis of the three-dimensional elasticity (Reddy, 1984; Grinchenko and Komissarova, 1974). In Table 1 they are also compared with the values derived from HSDPT [higher-order shear deformation plate theory (Reddy, 1984)].

Notice that apart from the case of simply-supported plates, it is rather difficult to obtain the exact values for free vibration frequencies from the three-dimensional elasticity; in Grinchenko and Komissarova (1974) for the case of a circular plate with a free edge only intervals containing the exact values were found.

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